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ON THE DIOPHANTINE EQUATION $x^4 + ay^4 = u^2 + bv^2$.

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§ 1. **Introduction. Diophantine equations of the general type**

$$\alpha x^m + \beta y^m = \gamma z^n \tag{1}$$

have been treated by several authors, notably by Desboves.[1] Here $\alpha, \beta, \gamma, m, n$ are given integers and integers x, y, z are to be determined so as to satisfy the equation. Recently, R. D. Carmichael[2] has given a partial treatment and has proposed a general investigation of the Diophantine equation

$$x^4 + ay^4 = u^2 + bv^2, \tag{2}$$

where a and b are given integers and integers x, y, u, v are to be determined so that equation (2) shall be satisfied. In case $a = 0$ or $b = 0$ the latter equation reduces to a special case of the former. The object of the present paper is to develop the theory of equation (2) in general and also for the four particular cases $a = 0$, $b = 0$, $b = a$, $b = \sigma^2 a$, where σ is an integer or the reciprocal of an integer. To a small extent (indicated by references below) there is a duplication of previous results; but the principal formulæ obtained are believed to be new. The methods employed throughout the paper are elementary.

§ 2. **Case when a = 0. The equation $\mathbf{u^2 + bv^2 = x^4}$.** The set of numbers $u^2 + bv^2$ forms a domain with respect to multiplication. This character is put in evidence by the easily verified formulæ:

$$(\alpha_1^2 + b\beta_1^2)(\alpha_2^2 + b\beta_2^2) = (\alpha_1\alpha_2 \pm b\beta_1\beta_2)^2 + b(\alpha_1\beta_2 \mp \alpha_2\beta_1)^2. \tag{3}$$

[1] *Nouvelles Annales de Mathématiques* (2), 18 (1879): 265–279, 398–410, 433–444, 481–499.

[2] R. D. Carmichael, *Diophantine Analysis*, pp. 46–48, 54.

In particular, we have

$$(\alpha^2 + b\beta^2)^2 = (\alpha^2 - b\beta^2)^2 + b(2\alpha\beta)^2. \tag{4}$$

Let $x = \alpha^2 + b\beta^2$. Then, by a repeated use of (4), we obtain

$$x^4 = [(\alpha^2 - b\beta^2)^2 - b(2\alpha\beta)^2]^2 + b[4\alpha\beta(\alpha^2 - b\beta^2)]^2. \tag{5}$$

From the last result we see that the equation

$$u^2 + bv^2 = x^4 \tag{6}$$

has the following two-parameter solution:

$$\begin{aligned} x &= \alpha^2 + b\beta^2, \\ u &= \alpha^4 - 6b\alpha^2\beta^2 + b^2\beta^4, \\ v &= 4\alpha\beta(\alpha^2 - b\beta^2). \end{aligned} \tag{7}$$

These formulæ are given by Desboves (l. c., p. 270).

Again, from (4) we may write

$$\begin{aligned} x^2 &= \alpha_1^2 + b\beta_1^2, \\ u &= \alpha_1^2 - b\beta_1^2, \\ v &= 2\alpha_1\beta_1. \end{aligned} \tag{8}$$

If we now treat similarly the first of equations (8), we will clearly be led to a solution of (6). This, however, will be the same as (7).

In order to obtain more general solutions of (6), let us consider the equation

$$x^2 + by^2 = \alpha_1^2 + b\beta_1^2. \tag{9}$$

Taking account of the double sign in the right-hand member of (3), we have readily the following solution of equation (9):

$$\begin{aligned} x &= s_1s_2 + bt_1t_2, \\ y &= s_1t_2 - s_2t_1, \\ \alpha_1 &= s_1s_2 - bt_1t_2, \\ \beta_1 &= s_1t_2 + s_2t_1. \end{aligned} \tag{10}$$

If we impose the condition that $s_1t_2 - s_2t_1 = 0$ or $s_1t_2 = s_2t_1$, we may obviously obtain a solution of (6) by combining equations (8) and (10). Put $s_1 = m - n$, $t_2 = m + n$, $s_2 = p - q$, $t_1 = p + q$. Then the condition $s_1t_2 = s_2t_1$ reduces to $m^2 - n^2 = p^2 - q^2$; and this is satisfied if

$$\begin{aligned} m &= \lambda_1\lambda_2 + \mu_1\mu_2, \\ n &= \lambda_1\mu_2 + \lambda_2\mu_1, \\ p &= \lambda_1\lambda_2 - \mu_1\mu_2, \\ q &= \lambda_1\mu_2 - \lambda_2\mu_1. \end{aligned} \tag{11}$$

Then we have

$$(12)\qquad \begin{aligned} s_1 &= \lambda_1\lambda_2 + \mu_1\mu_2 - \lambda_1\mu_2 - \lambda_2\mu_1, \\ s_2 &= \lambda_1\lambda_2 - \mu_1\mu_2 - \lambda_1\mu_2 + \lambda_2\mu_1, \\ t_1 &= \lambda_1\lambda_2 - \mu_1\mu_2 + \lambda_1\mu_2 - \lambda_2\mu_1, \\ t_2 &= \lambda_1\lambda_2 + \mu_1\mu_2 + \lambda_1\mu_2 + \lambda_2\mu_1. \end{aligned}$$

It is now clear that equations (8), (10) and (12) afford a four-parameter solution of (6).

In the special case when $\mu_1 = 0$ we have $s_1 = s_2$, $t_1 = t_2$. If we write $s_1 = s_2 = \alpha$, $t_1 = t_2 = \beta$, then the solution afforded by (8), (10), (12) becomes identical with solution (7). Hence (8), (10), (12) afford a solution which is a generalization of the previously known solution (7).

As illustrative examples of the solution of (6) afforded by (8), (10), (12), we have the following:

$b = 3$, $\lambda_1 = 2$, $\mu_1 = 1$, $\lambda_2 = 1$, $\mu_2 = 2$: $142^2 + 3\cdot 78^2 = 14^4$,

$b = 5$, $\lambda_1 = 2$, $\mu_1 = 1$, $\lambda_2 = 1$, $\mu_2 = 2$: $439^2 + 5\cdot 132^2 = 23^4$,

$b = 2$, $\lambda_1 = 3$, $\mu_1 = 1$, $\lambda_2 = 1$, $\mu_2 = 3$: $17^2 + 2\cdot 56^2 = 9^4$.

It should be noticed that the above solutions were not obtained merely by substitution. Factors common to u, v, x^2 have been removed after substitution in the formulæ.

§ **3. Case when b = 0. The equation $\mathbf{x^4 + ay^4 = u^2}$.** Consider the equation

$$(13)\qquad x^4 + ay^4 = u^2.$$

Let $u = \alpha^2 + a\beta^2$. Then $u^2 = (\alpha^2 - a\beta^2)^2 + a(2\alpha\beta)^2$. Equation (13) will therefore be satisfied if we write

$$(14)\qquad \begin{aligned} x^2 &= \alpha^2 - a\beta^2, \\ y^2 &= 2\alpha\beta, \\ u &= \alpha^2 + a\beta^2. \end{aligned}$$

The first of equations (14) is satisfied if

$$(15)\qquad \begin{aligned} x &= \alpha_1{}^2 - a\beta_1{}^2, \\ \alpha &= \alpha_1{}^2 + a\beta_1{}^2, \\ \beta &= 2\alpha_1\beta_1. \end{aligned}$$

Now, putting $\alpha_1 = \lambda^2$, $\beta_1 = \mu^2$ in (15) and combining equations (14) and (15), we have

$$(16)\qquad \begin{aligned} x &= \lambda^4 - a\mu^4, \\ y &= 2\lambda\mu\sqrt{\lambda^4 + a\mu^4}, \\ u &= \lambda^8 + 6a\lambda^4\mu^4 + a^2\mu^8. \end{aligned}$$

Hence, if λ and μ are values of x and y, respectively, satisfying (13), it is obvious that (16) will afford a second solution. Therefore, by a repeated use of (16), as many solutions as desired may be found.

Formulæ (16) were first obtained by Desboves (l. c., p. 437) by a method equivalent to that employed above.

From the second of equations (14) it follows that either α or β must contain 2 to an odd power. We shall first suppose it is α and write $\alpha = 2m\rho^2$, where m contains no repeated prime factor. Then, from the second equation (14) we see that we must have $\beta = m\sigma^2$. Now, substituting these values of α and β in the first equation (14), we see that values of ρ and σ satisfying the condition

$$4\rho^4 - a\sigma^4 = \tau^2 \tag{17}$$

will yield solutions of (13). If β contains 2 to an odd power, by reasoning similar to that employed above, we may write $\alpha = m\rho^2$, $\beta = 2m\sigma^2$. Making this substitution in the first equation (14), we see that values of ρ and σ satisfying the condition

$$\rho^4 - 4a\sigma^4 = \tau^2 \tag{18}$$

will yield solutions of (13).

Since equation (13) is known to have no solution when $a = 1$, we can not obtain a general solution free of restricting conditions. However, when $a = 2\lambda\mu(\lambda^2 + \mu^2)$, it can readily be shown that a solution is afforded by

$$\begin{aligned} x &= (\lambda - \mu)^2, \\ y &= 2(\lambda + \mu), \\ u &= \lambda^4 + 12\lambda^3\mu + 6\lambda^2\mu^2 + 12\lambda\mu^3 + \mu^4. \end{aligned} \tag{19}$$

When $a = 8\lambda\mu(\lambda^2 + \mu^2)$, it can readily be shown that a solution is afforded by

$$\begin{aligned} x &= (\lambda - \mu)^4 - 8\lambda\mu(\lambda^2 + \mu^2), \\ y &= 2(\lambda - \mu)(\lambda + \mu)^2, \\ u &= (\lambda - \mu)^8 + 48\lambda\mu(\lambda^2 + \mu^2)(\lambda - \mu)^4 + 64\lambda^2\mu^2(\lambda^2 + \mu^2)^2. \end{aligned} \tag{20}$$

Also, it is obvious that if a is a perfect square diminished by unity, integral solutions are possible. Solutions may be obtained when a has other special forms.

The following taken in order are examples illustrating the use of formulæ (16), (19), (20):

$$a = 3, \quad \lambda = 1, \quad \mu = 2: \quad 47^4 + 3\cdot 28^4 = 2593^2,$$

$$a = 20, \quad \lambda = 2, \quad \mu = 1: \quad 1^4 + 20\cdot 6^4 = 161^2,$$

$$a = 80, \quad \lambda = 2, \quad \mu = 1: \quad 79^4 + 80\cdot 18^4 = 6881^2.$$

§ 4. Case when b = a. The equation $x^4 + ay^4 = u^2 + av^2$. From formulæ (3) we see that the equation

$$x^4 + ay^4 = u^2 + av^2 \tag{21}$$

is satisfied if

$$(22)\qquad \begin{aligned} x^2 &= s_1s_2 - at_1t_2,\\ y^2 &= s_1t_2 + s_2t_1,\\ u &= s_1s_2 + at_1t_2,\\ v &= s_1t_2 - s_2t_1. \end{aligned}$$

Let $s_1 = \lambda - 3a\mu$, $s_2 = \lambda$, $t_1 = \lambda + a\mu$, $t_2 = -\mu$. Then equations (22) become

$$(23)\qquad \begin{aligned} x &= \lambda - a\mu,\\ y^2 &= \lambda^2 + (a-1)\lambda\mu + 3a\mu^2,\\ u &= \lambda^2 - 4a\lambda\mu - a^2\mu^2,\\ v &= 3a\mu^2 - (a+1)\lambda\mu - \lambda^2. \end{aligned}$$

It may be easily verified (cf. R. D. Carmichael, l. c., p. 25) that

$$(24)\qquad (m^2+amn+bn^2)^2 = (m^2-bn^2)^2+a(m^2-bn^2)(2mn+an^2)+b(2mn+an^2)^2.$$

Therefore, from the second of equations (23), we may write

$$(25)\qquad \begin{aligned} y &= m^2 + (a-1)mn + 3an^2,\\ \lambda &= m^2 - 3an^2,\\ \mu &= 2mn + (a-1)n^2. \end{aligned}$$

It is now evident that equations (23) and (25) afford a two-parameter solution of (21).

Again, make the following substitution in (22): $s_1 = \lambda - 3a\mu$, $s_2 = \lambda + a\mu$, $t_1 = a\mu$, $t_2 = \lambda$. We then obtain

$$(26)\qquad \begin{aligned} x^2 &= \lambda^2 - (a^2+2a)\lambda\mu - 3a^2\mu^2,\\ y &= \lambda - a\mu,\\ u &= \lambda^2 + (a^2-2a)\lambda\mu - 3a^2\mu^2,\\ v &= \lambda^2 - 4a\lambda\mu - a^2\mu^2. \end{aligned}$$

Making use of (24), we see that the first of equations (26) is satisfied if

$$(27)\qquad \begin{aligned} x &= m^2 - (a^2+2a)mn - 3a^2n^2,\\ \lambda &= m^2 + 3a^2n^2,\\ \mu &= 2mn - (a^2+2a)n^2. \end{aligned}$$

It is now clear that equations (26) and (27) afford a two-parameter solution of (21).

It should be observed that the above methods are capable of yielding an unlimited number of two-parameter solutions. Thus solutions may be obtained by making the following substitutions in (22):

$$\begin{array}{llll} s_1 = \lambda + 2a\mu, & s_2 = \lambda, & t_1 = -2\mu, & t_2 = \lambda + 2a\mu, \\ s_1 = \lambda + 3a\mu, & s_2 = \lambda, & t_1 = -\mu, & t_2 = \lambda + 4a\mu, \\ s_1 = \lambda - 8a\mu, & s_2 = \lambda, & t_1 = -4\mu, & t_2 = \lambda + a\mu, \\ s_1 = \lambda + 3a\mu, & s_2 = \lambda, & t_1 = -3\mu, & t_2 = \lambda + 3a\mu. \end{array}$$

From the above values of s_1, s_2, t_1, t_2 integers x are obtained by mere substitution. The corresponding expressions for y^2 contain λ^2 with coefficient unity. Similarly, the following substitutions yield y directly, the expressions for x^2 containing λ^2 with coefficient unity:

$$\begin{array}{llll} s_1 = \lambda, & s_2 = \lambda + a\mu, & t_1 = 4a\mu, & t_2 = \lambda, \\ s_1 = \lambda + 2a\mu, & s_2 = \lambda + 2a\mu, & t_1 = 2a\mu, & t_2 = \lambda, \\ s_1 = \lambda + 3a\mu, & s_2 = \lambda + 3a\mu, & t_1 = 3a\mu, & t_2 = \lambda, \\ s_1 = \lambda - 3a\mu, & s_2 = \lambda + a\mu, & t_1 = 9a\mu, & t_2 = \lambda. \end{array}$$

From the manner of formation of the expressions for x^2 and y^2, it is easy to see that as many substitutions as desired may be found such that one will be a perfect square and the other will contain λ^2 with coefficient unity.

The following taken in order are examples illustrative of the above methods:

$$a = 2,\quad m = 1,\quad n = 1,\quad \lambda = -5,\quad \mu = 3:\quad 11^4 + 2\cdot 8^4 = 109^2 + 2\cdot 74^2,$$

$$a = 2,\quad m = 1,\quad n = 1,\quad \lambda = 13,\quad \mu = -6:\quad 19^4 + 2\cdot 25^4 = 263^2 + 2\cdot 649^2.$$

§ 5. Case when $\mathbf{b = \sigma^2 a}$. The equation $\mathbf{x^4 + ay^4 = u^2 + \sigma^2 av^2}$. In order to develop the theory for this case we shall treat the more general equation

$$x^4 + ay^4 = u^2 + bv^2 \tag{28}$$

by means of more general formulæ than have been used in the preceding sections. This may be done by extending either of the sets of numbers $x^4 + ay^4 - bv^2$ and $u^2 + bv^2 - ay^4$ so as to form a multiplicative domain. Extending the latter, the required set of numbers is $u^2 + bv^2 - az^2 - abw^2$.

Consider the equation

$$u^2 + bv^2 - ay^4 - abw^2 = x^4. \tag{29}$$

If $g(u, v, y, w) = u^2 + \alpha v^2 + \beta y^2 + \alpha\beta w^2$, it can readily be verified (cf. R. D. Carmichael, l. c., p. 37) that

$$\begin{aligned} \{g(u, v, y, w)\}^2 &= g(u^2 - \alpha v^2 - \beta y^2 + \alpha\beta w^2,\ 2uv - 2\beta yw,\ 2yu + 2\alpha wv,\ 0), \\ &= g(u^2 - \alpha v^2 + \beta y^2 + \alpha\beta w^2,\ 2uv,\ 2\alpha wv,\ 2yv). \end{aligned} \tag{30}$$

Hence, using the first of equations (30), we see that (29) will be satisfied if

$$
(31)\qquad
\begin{aligned}
u &= u_1^2 - bv_1^2 + ay_1^2 - abw_1^2,\\
v &= 2u_1v_1 + 2ay_1w_1,\\
y^2 &= 2y_1u_1 + 2bw_1v_1,\\
w &= 0,\\
x^2 &= u_1^2 + bv_1^2 - ay_1^2 - abw_1^2.
\end{aligned}
$$

Suppose first that $b = m^2a$, where m is an integer. Also, put $u_1 = m(\lambda - a\mu)$, $v_1 = 2\lambda$, $y_1 = 2m(\lambda - a\mu)$, $w_1 = 4\mu$ in (31). Then

$$
(32)\qquad
\begin{aligned}
u &= m^2[\lambda^2 - (8a^2 + 2a)\lambda\mu + (4a^3 - 15a^2)\mu^2],\\
v &= 4m(\lambda^2 + 3a\lambda\mu - 4a^2\mu^2),\\
y &= 2m(\lambda + a\mu),\\
x^2 &= m^2[\lambda^2 + (8a^2 - 2a)\lambda\mu - (4a^3 + 15a^2)\mu^2].
\end{aligned}
$$

From (24) we see that the last of equations (32) is satisfied if

$$
(33)\qquad
\begin{aligned}
x &= m[\alpha^2 + (8a^2 - 2a)\alpha\beta - (4a^3 + 15a^2)\beta^2],\\
\lambda &= \alpha^2 + (4a^3 + 15a^2)\beta^2,\\
\mu &= 2\alpha\beta + (8a^2 - 2a)\beta^2.
\end{aligned}
$$

Equations (32) and (33) afford two-parameter solutions of equations of the general type

$$
(34)\qquad x^4 + ay^4 = u^2 + m^2av^2.
$$

Now, put $a = m^2b$, $u_1 = m(\lambda - b\mu)$, $v_1 = 2m^2\lambda$, $y_1 = 2m(\lambda - b\mu)$, $w_1 = 4\mu$ in (31). Then, after removing a factor m^2 common to x^2, y^2, u, v, we have

$$
(35)\qquad
\begin{aligned}
u &= \lambda^2 - (8m^2b^2 + 2b)\lambda\mu + (4m^2b^3 - 15b^2)\mu^2,\\
v &= 4m(\lambda^2 + 3b\lambda\mu - 4b^2\mu^2),\\
y &= 2(\lambda + b\mu),\\
x^2 &= \lambda^2 + (8m^2b^2 - 2b)\lambda\mu - (4m^2b^3 + 15b^2)\mu^2.
\end{aligned}
$$

Making use of (24) we see that the last of equations (35) is satisfied if

$$
(36)\qquad
\begin{aligned}
x &= \alpha^2 + (8m^2b^2 - 2b)\alpha\beta - (4m^2b^3 + 15b^2)\beta^2,\\
\lambda &= \alpha^2 + (4m^2b^3 + 15b^2)\beta^2,\\
\mu &= 2\alpha\beta + (8m^2b^2 - 2b)\beta^2.
\end{aligned}
$$

Equations (35) and (36) afford two-parameter solutions of equations of the general type

$$x^4 + m^2by^4 = u^2 + bv^2. \tag{37}$$

If we put $m = 1$ in (32), (33) and (35), (36), we are led to solutions of (21). These are identical.

As examples illustrating the use of the above methods, we have the following:

$b = m^2a,\ a = 1,\ m = 2,\ b = 4,\ \alpha = 2,\ \beta = 1:\quad 1^4 + 1\cdot 22^4 = 319^2 + 4\cdot 182^2,$

$a = m^2b,\ b = 1,\ m = 2,\ a = 4,\ \alpha = 2,\ \beta = 1:\ 11^4 + 4\cdot 46^4 = 4231^2 + 1\cdot 152^2.$

§ 6. **The general case. The equation $\mathbf{x^4 + ay^4 = u^2 + bv^2}$.** For the purpose of developing the theory for the general case we may employ equations (31). Put $y_1 = 0$, $v_1 = 2bw_1$ in (31). Then

$$\begin{aligned} u &= u_1^2 - (4b^3 + ab)w_1^2, \\ v &= 4bu_1w_1, \\ y &= 2bw_1, \\ x^2 &= u_1^2 + (4b^3 - ab)w_1^2. \end{aligned} \tag{38}$$

The last equation (38) is satisfied if

$$\begin{aligned} x &= u_2^2 + (4b^3 - ab)w_2^2, \\ u_1 &= u_2^2 - (4b^3 - ab)w_2^2, \\ w_1 &= 2u_2w_2. \end{aligned} \tag{39}$$

Equations (38) and (39) clearly afford a solution of (28).

Now, put $y_1 = 0$, $w_1 = 2bv_1$ in equations (31). They then become

$$\begin{aligned} u &= u_1^2 - (b + 4ab^3)v_1^2, \\ v &= 2u_1v_1, \\ y &= 2bv_1, \\ x^2 &= u_1^2 + (b - 4ab^3)v_1^2. \end{aligned} \tag{40}$$

The last of equations (40) will be satisfied if we write

$$\begin{aligned} x &= u_2^2 + (b - 4ab^3)v_2^2, \\ u_1 &= u_2^2 - (b - 4ab^3)v_2^2, \\ v_1 &= 2u_2v_2. \end{aligned} \tag{41}$$

Equations (40) and (41) afford a solution of (28).

Again, using the second formula (30), the last of equations (31) is satisfied if

$$
\begin{aligned}
x &= u_2^2 + bv_2^2 - ay_2^2 - abw_2^2,\\
u_1 &= u_2^2 - bv_2^2 - ay_2^2 - abw_2^2,\\
v_1 &= 2u_2v_2,\\
y_1 &= 2bw_2v_2,\\
w_1 &= 2y_2v_2.
\end{aligned}
\tag{42}
$$

Put $y_2 = w_2 = bv_2$. Then, from the third equation in (31), we have the further restriction

$$
y^2 = 4b^2v_2^2[u_2^2 + 2u_2v_2 - (b + ab^2 + ab^3)v_2^2],
\tag{43}
$$

which is satisfied in accordance with (24) if

$$
\begin{aligned}
y &= 2bv_2[m^2 + 2mn - (b + ab^2 + ab^3)n^2],\\
u_2 &= m^2 + (b + ab^2 + ab^3)n^2,\\
v_2 &= 2mn + 2n^2.
\end{aligned}
\tag{44}
$$

It will now be seen that equations (31), (42), (44) afford a solution of (28).

The following taken in order are examples illustrative of the above methods:

$a=5,\ b=2,\ u_2=1,\ w_2=1,\ u_1=-21,\ w_1=2$: $23^4+5\cdot 8^4 = 273^2+2\cdot 336^2$,

$a=3,\ b=2,\ u_2=1,\ v_2=1,\ u_1=95,\ v_1=2$: $93^4+3\cdot 8^4 = 8633^2+2\cdot 380^2$,

$a=3,\ b=2,\ n=1,\ u=1,\ u_2=39,\ v_2=4$: $977^4+3\cdot 560^4=663457^2+2\cdot 618864^2$.

If we operate on the last of equations (31) by means of the first formula (30) and make certain easy transformations, we will be led at once to the following solution of (28):

$$
\begin{aligned}
x &= \alpha^4 - a\beta^4 + b\gamma^2,\\
y &= 2\alpha\beta\sqrt{\alpha^4 + a\beta^4 - b\gamma^2},\\
u &= (\alpha^4 + a\beta^4 - b\gamma^2)^2 - 4b\alpha^4\gamma^2 + 4a\alpha^4\beta^4,\\
v &= 4\alpha^2\gamma(\alpha^4 + a\beta^4 - b\gamma^2).
\end{aligned}
\tag{45}
$$

These formulæ have been developed in full by R. D. Carmichael (l. c., pp. 46–47) and will therefore be dismissed with this remark.